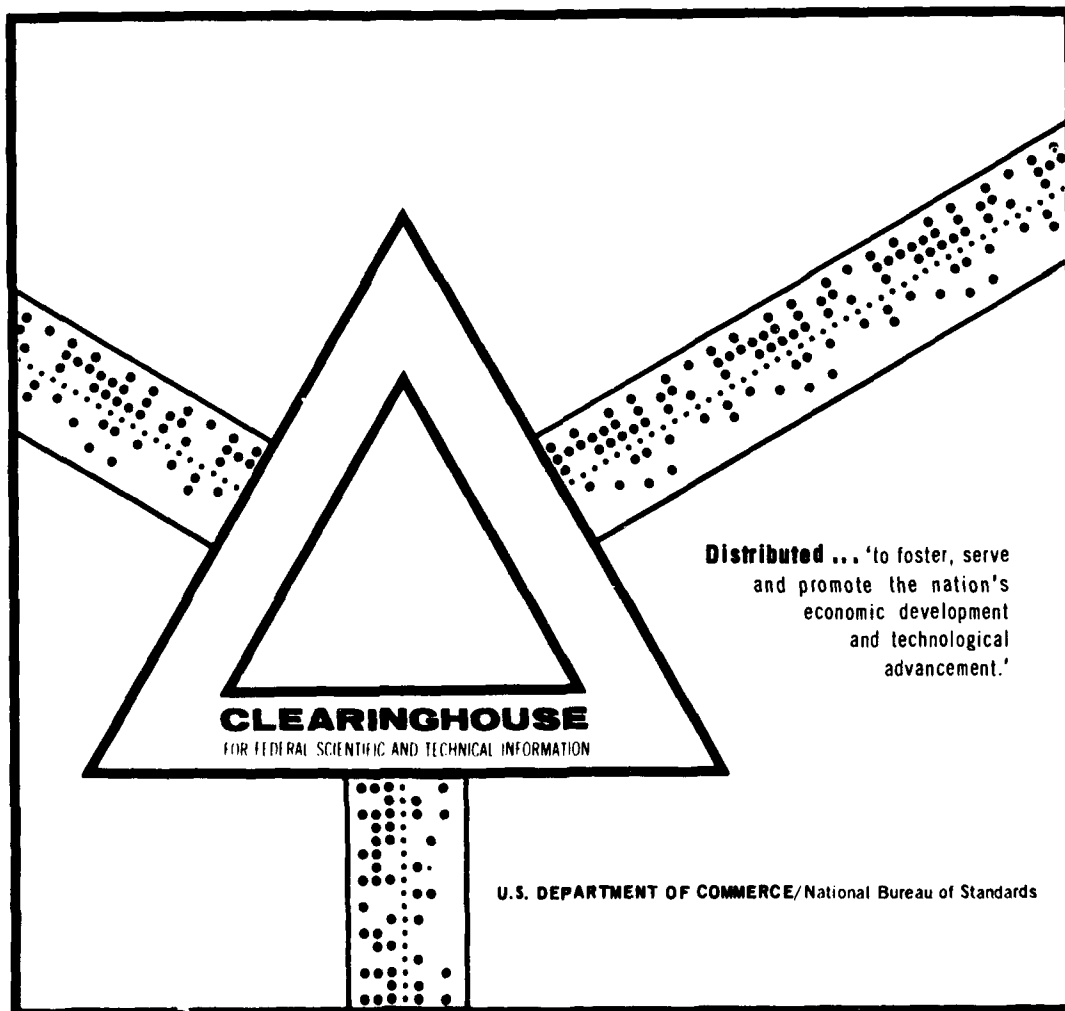SMOOTHNESS OF SOLUTIONS OF DEGENERATING ELLIPTIC
EQUATIONS

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SMOOTHNESS OF SOLUTIONS OF DEGENERATING ELLIPTIC EQUATIONS

by

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translated from

by

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Summary

In this study it is proven that the generalized solution
of boundary-value problems for degenerating second-order
equations satisfies Holder's boundary condition under broad
assumptions. An example is given, showing that a greater
smoothness of the solutions cannot be achieved only by in-
creasing the smoothness of the data given in the problem.
Conditions for the existence of derivatives of the general-
ized solution are clarified. The investigation is carried
out by means of probability methods.

39

SMOOTHNESS OF SOLUTIONS OF DEGENERATING ELLIPTIC EQUATIONS*

by

M. F. Freidlin

Introduction

The present study is concerned with boundary-value problems for the degenerating elliptic equation

$$Lu(x) + c(x)u(x) = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} +$$
$$+ \sum b_i(x) \frac{\partial u}{\partial x_i} + c(x)u(x) = 0. \tag{1}$$

It is assumed that the coefficients are defined over all spaces R_n, are bounded and have bounded first-order derivatives

$$\max_{1 \le i, j, k \le n, x \in R^n} \left\{ \left| \frac{\partial a_{ij}(x)}{\partial x_k} \right|, \left| \frac{\partial b_i(x)}{\partial x_k} \right|, \left| \frac{\partial c(x)}{\partial x_k} \right| \right\} < K < \infty;$$

the form $\sum_{i,j=1}^{n} a_{ij}(x)\lambda_i\lambda_j$ is considered non-negative. The study of boundary-value problems for degenerating equations has been the object of numerous investigations. References [1] and [11] can be mentioned, where equations degenerating on the boundary of a region are studied. Special cases of degeneration inside a region are examined in references [12-15]. The boundary-value problem for an equation with degeneration of a general form was examined for the first time by Fichera (see [4,5]). However, a single class of functions, in which the problem would have a unique generalized solution, was not constructed in these studies. In a subsequent study [12], a uniqueness class was constructed and conditions were

established for equations with degeneration of a rather special form, under which a generalized solution will be smooth. In the general case, the existence and uniqueness of a generalized solution in one class were proven in [17] and [18] (see also [9]). Note that in Fichera's study [4] and others associated with it the assumption is made that $c(x) < - c_o < 0$. This assumption removes certain interesting effects on the boundary (pasting part of the boundary) at one point [11] and makes the problem of uniqueness more simple. If this assumption is rejected, in order to ensure the uniqueness of the generalized solution it is necessary, in general, to prescribe boundary conditions on a certain "inner boundary" (see [22]). This "inner boundary" arises naturally when stabilization of the solution of the mixed problem is studied as $t \to \infty$.

In [20] we find conditions which guarantee that the generalized solution, in the case of degeneracy of a general form, is smooth and satisfies the Hölder condition. Proof of the statements formulated in [20] constitutes precisely the basic content of the present study. The smoothness of generalized solutions was also studied in references [9] and [10].

Generally speaking, the generalized solution of the Dirichlet problem for equation (1) will be discontinuous. It was found that, under very general assumptions in regard to the coefficients and the boundary function, the Hoelder character of the generalized solution can be guaranteed (see example and Theorem 1). In order that the solution should have derivatives, it is no longer sufficient to require only that the data of the problem be smooth, but it is also necessary to impose inequality-type conditions on the coefficients and on their first derivatives. In

order to obtain a priori estimates which ensure smoothness of the generalized
solution we shall use a representation of the solution of the Dirichlet problem
as the mean value of a certain functional of trajectories of a random Markov pro-
cess controlled by the operator L (see [2], [8]). We believe that such an approach
to the problem is a geometrically descriptive and natural one. In particular, the
conditions of the existence of a priori estimates of derivatives of the generalized
solution acquire a simple geometrical meaning. In the present study we shall use
the same notation as that used in reference [18].

1. Generalized Solution of the Dirichlet Problem

By $\sigma(x)$ we shall denote a matrix such that $\{a_{ij}(x)\} = a(x) = \sigma(x)\,\sigma^*(x)$ (the
star means transposition). Let us assume that the matrix $\sigma(x)$ can be chosen in
such a way that all its elements are differentiable, and let

$$\max \left| \frac{\partial \sigma_{ij}(x)}{\partial x_h} \right| < K.$$

According to the condition, $a(x)$ is a symmetric positive-definite matrix, and there-
fore we will always find a $\sigma(x)$ satisfying the relation $\sigma(x)\,\sigma^*(x) = a(x)$. Our
assumptions refer only to the smoothness of the functions $\sigma_{ij}(x)$ with respect to x.
This problem was investigated in [25], where it was shown, in particular, that if
$a_{ij}(x) \in C^2(R^n)$ then, regardless of the form of degeneracy, the matrix $\sigma(x)$ can
always be chosen in such a way that its elements satisfy the Lipschitz condition,
where the Lipschitz constant is estimated in terms of the norm $a_{ij}(x)$ in $C^2(R^n)$.

If the rank of $\{a_{ij}(x)\}$ is constant, the elements of $\sigma(x)$ have the same
smoothness as the functions $a_{ij}(x)$.

We shall examine the stochastic differential equation (see [6,8])

$$dx_t(\omega) = \sigma(x_t)d\xi_t(\omega) + b(x_t)dt, \tag{2}$$

where $\xi_t(\omega)$ is an n-dimensional Wiener process (see [2], $b(x) = (b_1(x),\ldots,b_n(x))$. This equation, together with the initial condition $x_0(\omega) = x \in R^n$, has the unique solution $x_t^x(\omega)$, which defines the Markov process $X = \{x_t, P_x\}$ (see [2]). The process X is said to be governed by operator L. The properties of solutions of boundary-value problems for operator L are closely associated with the behavior of the trajectories of process X.

Let D be a bounded region in R^n with the boundary Γ. The point $x_0 \in \Gamma$ will be called regular if

$$\lim_{x \to x_0} P_x\{x_{\tau_\varepsilon} \in \Gamma\} = 1$$

for any $\varepsilon > 0$, where τ_ε is the moment at which the trajectory of process X first reaches the boundary of the set $U_\varepsilon(x_0) \cap D$. Here, as well as in what follows, we denote by $U_\varepsilon(x_0)$ the ε-neighborhood of the point x_0. We can prove the following lemma (see [11]):

Lemma 1. In order that point $x_0 \in \Gamma$ be regular for the process X, controlled by operator L, it is sufficient that there exist a function v(x) (barrier) such that v(x) is continuous in a certain neighborhood U of the point x_0 and that $v(x_0) = 0$, $v(x) > 0$ and $Lv(x) \leq 0$ for $x \in U\backslash x_0$.

The point x_0 will be called (λ_1,λ_2,h)-regular if there exists a function v(x) with properties specified in lemma 1 such that

$$c_1|x - x_0|^{\lambda_1} \leq v(x) \leq c_2|x - x_0|^{\lambda_2} \text{ for } x \in U_h(x_0).$$

The following lemma gives sufficient conditions of (λ_1,λ_2,h)-regularity in terms of the coefficients.

Lemma 2. **Let us assume that at least one of the following conditions is fulfilled:**

1) **the coefficients of the operators and the direction cosines** $\{n_i(x)\}$ **of the normal to** Γ **belong to class** C^2 **in the neighborhood of the point** x_o **and**

$$\sum_{i,j=1}^{n} a_{ij}(x_0)n_i(x_0)\times n_j(x_0) > 0;$$

2) **the point** x_o **can be touched by the half-space from outside the region** D **and** $(b(x_o),n(x_o)) > 0$, **where** $n(x_o)$ **is the outward normal to the support hyperplane.**

Then, **the point** $x_o \in \Gamma$ **is** (λ_1,λ_2,h)-**regular for some** $\lambda_1,\lambda_2,h > 0$.

Proof. Let the first condition be fulfilled. Let us introduce the co-ordinates $x_1',\ldots,x_n'$ in such a way that the boundary of region D in the neighborhood of x_o be described by equation $x_o' = 0$ and that region D lie in the half-space $x_1' < 0$. In this coordinate system $a_{11}'(x_o) > 0$. The function $v(x)$ can be chosen in the form

$$v(x) = -ax_1' - bx_1' + c\sum_{i=2}(x_i')^2.$$

It is easy to verify that the function $v(x)$ is the barrier being sought in case of a proper selection of coefficients a,b,c in a sufficiently small neighborhood of the point x_o.

If condition 2) is fulfilled, we perform a linear transformation of variables in which the half-space, by means of which the point x_o can be touched from the

outside, will be described by the equation $x_1' = 0$. Since in the case of linear transformation of independent variables $b(x) = \{b_1(x),\ldots,b_n(x)\}$ is transformed as a vector, we have $b_1'(x_o) > 0$. In this case, the function $v(x)$ can be chosen in the form

$$v(x) = -ax_1' + b \sum_{i=2}^{n} (x_i')^2.$$

Note that in both cases $\lambda_1 = \lambda_2 = 1$.

Remark 1. If the boundary and the coefficients in the neighborhood of point x_o are smooth, the requirement that point x_o can be touched by a half-space from outside the region D can be replaced by the assumption that point x_o belongs to the closure of the set open on Γ, in which

$$\sum_{i,j=1}^{n} a_{ij}(x) n_i(x) n_j(x) = 0.$$

The point $x_o \in \Gamma$ is called normally regular if $M_x \tau \leq |x - x_o|$, where τ is the instant of the first exit from region $D : \tau = \inf \{t : x_t \in D\}$.

Lemma 3. _If the conditions of lemma 2 are fulfilled, then the point x_o is normally regular._

For the proof we can make use of the barriers introduced in lemma 2. Indeed, if we denote by α the characteristic operator of process X (see [2]), we have

$$\mathfrak{A} M_x \tau = -1, \quad M_x \tau |_{x \in \Gamma} = 0, \quad \mathfrak{A} v(x) = L v(x) = -c < 0.$$

Then $\dfrac{A}{c} v(x) - M_x \tau \geq 0$ on the boundary of region $U_h(x_o)$ for sufficiently large A and

$$\mathfrak{A} \left(\frac{A}{c} v(x) - M_x \tau \right) = -A + 1.$$

Consequently, by virtue of the maximum principle

$$M_x\tau < \frac{A}{c}v(x) < \tilde{c}_1|x - x_0|.$$

If $M_x\tau < c_1'|x - x_0|$ for any point $x_0 \in \gamma \subset \Gamma$, we shall call the set γ uniformly normally regular.

The set $\gamma \subset \Gamma$ is called repelling if

$$P_x\{\varliminf_{t \to \tau} \rho(x_t, \gamma) > 0\} = 1$$

for any $x \in D$, $\tau = \inf \{t : x_t \bar{\in} D\}$. It is possible to give sufficient conditions in order that the set $\gamma \subset \Gamma$ be repelling in terms of the coefficients (see [11]).

Let us recall the definition of the generalized solution (see [18]). Let $\widetilde{X} = \{\widetilde{x}_t, \widetilde{P}_x\}$ be the Markov process obtained on X by a stop on the boundary Γ of region D, and let $\widetilde{T}_t$ be the semi-group of operators which acts in the space B of bounded measurable functions on $D \cup \Gamma$ according to the formula

$$T_t f(x) = \widetilde{M}_x f(\widetilde{x}_t).$$

By $\widetilde{\Gamma}$ we shall denote a part of the boundary Γ of region D such that $\Gamma \backslash \widetilde{\Gamma}$ is a repelling set. We shall assume that all points of $\widetilde{\Gamma}$ are regular.

As a generalized solution of the problem

$$Lu(x) = 0, \quad u(x)|_{x \in \widetilde{\Gamma}} = \psi(x) \tag{3}$$

we shall call the function $u(x)$, which assumes boundary values at points of $\widetilde{\Gamma}$ where

$\psi(x)$ is continuous and which satisfies the relation (see Note) $\widetilde{T}_t u(x) = u(x)$ for all $t \geq 0$. The correctness of such a definition is proven in [18].

It can be verified (see, for example, [18]) that the function $u(x) = M_x \psi(x_\tau)$ ($\tau = \inf \{t : x_t \bar{\in} D\}$) satisfies the relation $\widetilde{T}_t u = u$ and boundary conditions on $\widetilde{\Gamma}$. Without additional assumptions, this solution, generally speaking, is not unique.

It can be proven (see [18]) that for uniqueness it is necessary and sufficient that $P_x \{\tau = \infty\} = 0$. Everywhere in what follows, we assume the fulfillment of a somewhat stronger condition of a uniformly rapid exit of the trajectories onto the boundary:

$$\lim_{t \to \infty} P_x \{\tau > t\} = 0$$

uniformly with respect to $x \in D$. The latter equality will be fulfilled, for example, if at least one coefficient of operator L is different from zero in the entire region $D \cup \Gamma$ (see [18]).

Lemma 4. If the trajectories of process X leave region D with a uniform speed, then an $\alpha_{L,D} > 0$ can be found such that

$$P_x \{\tau > t\} < ce^{-\alpha_{L,D} t}$$

for any $x \in D$.

Proof. Since the trajectories reach the boundaries at a uniformly rapid speed, for a certain t_0 we have, for all $x \in D$, $\quad P_x \{\tau > t_0\} \leq \beta < 1$.

Note: This relation is equivalent to the equality $Au(x) = 0$, where A is an infinitesimal operator of the semi-group T_t.

We denote

$$a(n) = \sup_{x \in D} P_x \{\tau > nt_0\}.$$

(4)

Using the Markov property of process X, we get

$$P_x\{\tau > nt_0\} = M_x \chi_{\tau > t_n}(\omega) \cdot M_{x_{t_0}} \chi_{\tau > t_0(n-1)} \leqslant$$
$$\leqslant (M_x \chi_{\tau > t_n} \cdot M_x (M_{x_{t_0}} \chi_{\tau > t_0(n-1)})^2)^{1/2} \leqslant$$
$$\leqslant (P_x \{\tau > t_0\} \cdot a^2(n-1))^{1/2} \leqslant \sqrt{\beta} a(n-1),$$

where $\chi_{\tau > s}(\omega)$ is the characteristic function of the set $\{\omega : \tau(\omega) > s\}$. From (4) we conclude that $a(n) < \sqrt{\beta} a \, (n - 1)$, consequently $a(n) < e^{n \ln \sqrt{\beta}}$. From the latter inequality follows the statement of the lemma for

$$\alpha_{L,D} = -\frac{1}{2} \ln \beta > 0.$$

Corollary. If the conditions of lemma 4 are fulfilled, the random variable τ has all moments and

$$M_x \tau^k < c \int_0^\infty x^k e^{-\alpha_{L,D} x} dx, \quad M_\lambda \exp\{\lambda \tau\} < \infty \text{ for } \lambda < \alpha_{L,D}.$$

It is not difficult to find the lower bound for the constant $\alpha_{L,D}$ in terms of the coefficients of the operator and the dimensions of region D. For example, if the diameter of region D is equal to d, $b_1(x) > b > 0$, $0 \leq a_{11}(x) < a < \infty$, then

$$P_x\{\tau > t\} \leqslant P_x \left\{ \int_0^t \sum_{i=1}^n \sigma_{1i}(x_s) \, d\xi_s^i > bt + d \right\} \leqslant$$
$$\leqslant P \left\{ \xi_{\int_0^t a_{11}(x_s) ds} > bt + d \right\} \leqslant P \left\{ \xi_{at} > \frac{bt + d}{2} \right\} \leqslant c e^{-\frac{b^2}{4a} t}.$$

Note that in this case $\alpha_{L,D}$ is estimated by a constant which does not depend on the diameter of the region.

If $a_{11}(x) > a > 0$ everywhere in D and $\left|b_1(x)\right| < b < \infty$, then, during the time $t = 1$, the trajectories originating from any point $x \in D$ leave the region D with a probability greater than $\beta = P\{\xi_a > b + d\} = P\{\xi_1 > \frac{b+d}{\sqrt{a}}\}$. Hence it follows (see proof of lemma 4) that

$$\alpha_{L,D} \geqslant -\frac{1}{2} \ln \beta.$$

If, for a certain i, $a_{ii}(x) \equiv 0$ everywhere in region D, $b_i(x) \neq 0$ in $D \cup \Gamma$, and region D lies between the planes $x_i = a$, $x_i = a + h$, then $\alpha_{L,D} = \infty$. This follows from the fact that, in this case, a determinate motion takes place along the axis x_i with a nonvanishing speed.

In concluding this section we shall note that in the nondegenerate case the limit

$$\lim_{t \to \infty} \frac{1}{t} \ln P_x^{-1} \{\tau > t\}$$

exists, does not depend on x and is equal to the first eigenvalue of the problem. In the degenerating case,

$$\lim_{t \to \infty} \frac{1}{t} \ln P_x^{-1} \{\tau > t\}$$

has a similar meaning but, for sufficiently strong degeneration, this limit depends on the initial point x. The assumption that trajectories leave the region with a uniform speed ensures the inequality

$$\inf_{x \in D} \lim_{t \to \infty} \frac{1}{t} \ln P_x^{-1} \{\tau > t\} > 0.$$

2. Example. Idea of the Proof

In this section we shall give an example showing that the estimates of the smoothness of the generalized solution, which will be presented in subsequent sections, cannot be improved substantially. Using this example, it is also possible to grasp those ideas which are used in studying smoothness in a general case.

Let region D be a square: $D = \{x,y: |x| < 1, |y| < 1\}$. By $\varphi(x,y)$ let us denote an infinitely differentiable function, which is even with respect to y and equal to zero outside the ϵ-neighborhood of the boundary Γ of region D. Let us examine the Dirichlet problem

$$lu = \alpha \frac{\partial^2 u}{\partial x^2} + \beta y \frac{\partial u}{\partial y} + \varphi^2(x, y) \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = 0,$$
$$u(x, y)\,|\,\Gamma \equiv y. \tag{5}$$

Operator ℓ does not degenerate in the neighborhood of the boundary, and the co-efficient $\alpha > 0$ in $D \cup \Gamma$; therefore, problem (5) has a unique generalized solution. This solution is continuous in the closed region $D \cup \Gamma$ (see [18]). Since operator ℓ does not change when y is replaced by -y, and the boundary function is multiplied by -1, we have $u(x,y) - -u(x,-y)$ and $u(x,0) - u(0,0) = 0$. From the maximum principle and the regularity of boundary points it follows that sign $u(x,y)$ = sign y and $u(x,y) > \frac{99}{100}$ $(< - \frac{99}{100})$ for $1 - y \leq \epsilon$ (when $1 + y \leq \epsilon$) and for a sufficiently small $\epsilon > 0$.

By $z_t = (x_t, y_t)$ let us denote the trajectories of the Markov process in R^2 controlled by operator ℓ and by D_ϵ the region obtained from D by deleting the

ε-neighborhood of the boundary. The trajectories z_t can be constructed with the aid of the stochastic equations

$$x_t - x = \int_0^t \sqrt{\alpha + \varphi^2(x_s, y_s)}\, d\xi_s^1, \quad y_t - y = \int_0^t \varphi(x_s, y_s)\, d\xi_s^2 + \int_0^t \beta y_s\, ds, \tag{6}$$

where ξ_s^1, ξ_s^2 are independent Wiener processes and (x,y) is the initial point. The solution $u(x,y)$ of problem (5) satisfies the relation $u(x,y) = M_{(x,y)}\, u(z_{\tau_\varepsilon})$, where τ_ε is the instant of the first exit of trajectories z_t from the region D_ε. Let $y_o > 0$. Since $P_{x,y_o}\{y_{\tau_\varepsilon} > 0\} = 1$, we have

$$u(0, y_0) = M_{(0, y_0)} u(z_{\tau_\varepsilon}) \geqslant \frac{99}{100} P_{0, y_0}\{y_{\tau_\varepsilon} = 1 - \varepsilon\}. \tag{7}$$

Let us estimate the expression on the right-hand side of inequality (7). The motion along axis y in D_ε is determinate. Integrating equation $\frac{dy}{dt} = \beta y$ with the initial condition $y(0) = y_o$ we find that the time

$$t(y_0) = \ln\left(\frac{1 - \varepsilon}{y_0}\right)^{1/\beta}$$

is required in order to reach the point $y = 1 - \varepsilon$.

Note that

$$P_{(0, y_0)}\{y_{\tau_\varepsilon} = 1 - \varepsilon\} = P_{(0, y_0)}\{\sup_{s < t(y_0)} |x_s| < 1 - \varepsilon\}.$$

Since x_s in D_ε is a one-dimensional Markov process with the derivative operator $\alpha \frac{d^2}{dx^2}$, the function

$$v(t, x) = P_x\{\sup_{s < t} |x_s| < 1 - \varepsilon\}$$

is the solution of the mixed problem

$$\frac{\partial v}{\partial t} = \alpha \frac{\partial^2 v}{\partial x^2}, \quad v(t, x)\big|_{|x| = 1 - \varepsilon} = 0, \quad v(0, x) = 1.$$

Solving this problem (for example, by Fourier's method) we obtain that

$$v(0, t) = c_1 \exp\left\{-\frac{\alpha \pi^2 t}{(1 - \varepsilon)^2}\right\}$$

- 12 -

and, consequently,

$$P_{(0, y_0)} \left\{ \sup_{s < t(y_0)} |x_s| < 1 - \varepsilon \right\} = v(t(y_0), 0) =$$

$$= c_1 \exp \left\{ -\frac{a\pi^2}{(1-\varepsilon)^2} \ln \left(\frac{1-\varepsilon}{y_0} \right)^{1/\beta} \right\} = c_2 y_0^{\frac{a\pi^2}{\beta(1-\varepsilon)^2}},$$

where c_1 and c_2 do not depend on y_0. Comparing the equality obtained with (7) we conclude that

$$u(0, y_0) > c_2 \cdot \frac{99}{100} \cdot y_0^{\frac{a\pi^2}{\beta(1-\varepsilon)^2}}.$$

From the latter inequality, if we take into account that $u(0,0) = 0$, it follows that the function $u(x,y)$ may not have derivatives at the point $(0,0)$ without additional assumptions concerning the magnitude of α/β, and that for any $\gamma > 0$ it is possible to find an α/β so small that the function $u(x,y)$ will not satisfy Hölder's condition with the index γ.

We have introduced the term $\varphi^2(x,y)\Delta$ into the operator ℓ only in order to exclude the possibility of disturbing the smoothness at the expense of degeneration on the boundary. If we set $\varphi(x,y) \equiv 0$, all estimates for $|u(0,y) - u(0,0)|$ will remain in this case and it is only necessary to substitute $\varepsilon = 0$ into them.

We now obtain the upper bound for $|u(0,y_0) - u(0,0)|$. Since, for $\varphi(x,y) = 0$, motion along axis y is determinate in the entire region D, we have

$$|u(0, y_0) - u(0,0)| = |u(0, y_0)| = M_{0, y_0} |y_{\tau_D}| \leqslant$$

$$\leqslant M_{0, y_0} |y_\tau| = \int_0^\infty y_t(y_0) p_\tau(t) dt, \tag{8}$$

where $y_t(y_0)$ is the solution of equation $dy/dt = \beta y$ with the condition $y(0) = y_0$, while $p_\tau(t)$ is the density of the random variable $\tau = \inf\{t : |x_t| = 1\}$, starting

from the point x = 0, $\tau_D = \inf\{t : z_t \bar{\in} D\}$. This density exists since x_t differs from the Wiener process only by a factor. From the formulas given earlier it follows that

$$y_t(y_0) = y_0 \exp\{\beta t\}, \quad p_\tau(t) = -\frac{\partial v}{\partial t}(0, t) = c_2 e^{-\alpha \pi^2 t}.$$

Substituting these formulas into (8) we get

$$|u(0, y_0) - u(0, 0)| \leqslant c_3 y_0 \int_0^\infty e^{\beta t - \alpha \pi^2 t}\, dt.$$

If $\beta - \alpha \pi^2 < 0$, the integral on the right-hand side of the last inequality converges and the function $u(x, y)$ at the point $(0,0)$ satisfies the Lipschitz condition with respect to y. If $\beta - \alpha \pi^2 > 0$, then $u(x, y)$ does not satisfy at zero Hölder's condition with any index $\varkappa < \min(1, \frac{\alpha \pi^2}{\beta})$. Indeed, since $|y_{\tau_D}| < 1$, we will get, for $\varkappa < 1$

$$|u(0, y_0) - u(0, 0)| = M_{(0, y_0)}|y_{\tau_D}| \leqslant M_{(0, y_0)}|y_{\tau_D}|^\varkappa \leqslant$$

$$\leqslant M_{(0, y_0)}|y_\tau|^\varkappa = \int_0^\infty y_t^\varkappa(y_0) p_\tau(t)\, dt = c_3 y_0^\varkappa \int_{\bullet}^\infty e^{(\varkappa \beta - \alpha \pi^2) t}\, dt \leqslant c_2 c_4 y_0^\varkappa,$$

where

$$c_4 = \int_0^\infty e^{t(\varkappa \beta - \alpha \pi^2)}\, dt < \infty \quad \text{for } \varkappa < \frac{\alpha \pi^2}{\beta}.$$

Thus,

$$c_2 y_0^{\frac{\alpha \pi^2}{2}} < |u(0, y_0) - u(0, 0)| < c_3 y_0^\varkappa$$

for any

$$\varkappa < \min\left(1, \frac{\alpha \pi^2}{\beta}\right).$$

Let us note those qualitative considerations which were used in the above example and which can be carried over to the general case. We consider that $\frac{1}{\cdot}(x, y) \equiv 0$.

1. Let us consider the two trajectories $z_t(x_1,y_1)$ and $z_t(x_2,y_2)$, originating respectively from the points (x_1,y_1) and (x_2,y_2), for the same Wiener trajectories (ξ_t^1,ξ_t^2). The trajectories z_t are constructed with respect to (ξ_t^1,ξ_t^2) with the aid of equations (6). Then, after the time t the difference

$$x_t(x_1,y_1,\omega) - x_t(x_2,y_2,\omega) = x_1 - x_2$$

does not increase, while

$$|y_t(x_1,y_1,\omega) - y_t(x_2,y_2,\omega)| = |y_1 - y_2|e^{\beta t}$$

and, consequently,

$$|z_t(x_1,y_1,\omega) - z_t(x_2,y_2,\omega)|^2 = |x_1 - x_2|^2 + |y_1 - y_2|^2 e^{2\beta t}$$

grows at an exponential rapid rate with the index 2β. In the general case, the trajectories of the diffusion process with coefficients satisfying the Lipshitz condition, constructed by one and the same Wiener process and originating from the points a, b $\in$ R^n, resp., disperse at not greater than an exponential rate

$$M|z_t^a(\omega) - z_t^b(\omega)|^2 < |a - b|^2 e^{cKt},$$

where the constant K is determined from the Lipschitz constant of the coefficients (see lemma 5). In our case c = 2, K = β.

2. If by τ we denote the time needed by process Z to reach the boundary of region D, then

$$P_x\{\tau > t\} = f_x(t) < c_t e^{-\alpha \pi^2 t}$$

decreases at an exponential rate. In the general case, if the assumption that the trajectories reach the boundary at a uniform speed is fulfilled, this probability also decreases exponentially (see lemma 4).

3. We shall explain our further course of reasoning by means of an example describing how an estimate for the first derivative is obtained. In order not to be concerned with boundary estimates, we shall assume that the process does not

degenerate in the neighborhood of the boundary. Let

$$\bar{\tau} = \min(\tau_x, \tau_y), \quad \tau_x(\omega) = \inf\{t : x_t^x(\omega) \overline{\in} D\},$$

$$\tau_y(\omega) = \inf\{t : x_t^y(\omega) \overline{\in} D\}.$$

Then, using the probability representation of the solution of Dirichlet's problem, we will get

$$|u(x) - u(y)|^2 \leqslant M |u(x_{\bar{\tau}}^x) - u(x_{\bar{\tau}}^y)|^2 \leqslant cM |x_{\bar{\tau}}^x - x_{\bar{\tau}}^y|^2,$$

where c is a constant depending on the Lipschitz constant for the function u(x) in the neighborhood of the boundary and on the width of the non-degeneracy region. In order to obtain the estimate of the first derivative of interest to us, it is sufficient to show that $M|x_{\bar{\tau}}^x - x_{\bar{\tau}}^y|^2 < A |x - y|^2$. If the non-random variable t were present instead of the random variable τ, such an estimate would be guaranteed by item 1 above. In order to obtain an estimate for a random τ, we must use the properties given in items 1 and 2. If the exponent derived in item 2 "exceeds" the exponent of point 1 (in our example, $\alpha \pi^2 > \beta$) the estimate being sought exists. Otherwise, there may be no estimate. In the following sections we will obtain a priori estimates according to the plan outlined here.

In concluding this section we shall note that if the coefficients of the equation with degenerations satisfy only Hölder's condition, the generalized solution does not have to be continuous, even if the operator does not degenerate near the boundary and the statement of lemma 4 is fulfilled. An example confirming this remark can be obtained if we examine the equation

$$a \frac{\partial^2 u}{\partial x^2} + \varphi^2(x, y)\Delta u + \beta y^\gamma \frac{\partial u}{\partial y} = 0$$

in the square D. In this case, the trajectories scatter faster than exponentially, and an exponentially rapid exit from the region cannot "balance" this scattering.

3. Continuity of the Hölder-generalized solution

Theorem 1. Let us assume that all points of the set $\tilde{\Gamma}$ are $(\lambda_1, \lambda_2, h)$-regular for certain $\lambda_1, \lambda_2, h > 0$, which are the same for all $x \in \tilde{\Gamma}$. In addition, let the trajectories of process X go out at a uniform speed to the boundary of region D and let the boundary function satisfy Hölder's condition on Γ:

$$|\psi(x) - \psi(y)| < K_1 |x - y|^\nu.$$

Then there exists a $\gamma_1 > 0$ such that the generalized solution of problem (3) satisfies Hölder's condition with the index γ_1.

Proof. Let $x, y \in D$ and let $\bar{x}_t$ and $\bar{y}_t$ be the solutions of equations (2) with the initial conditions $\bar{x}_0 = x$, $\bar{y}_0 = y$. We set

$$\tau_x = \inf\{t: \bar{x}_t \overline{\in} D\}, \quad \tau_y = \inf\{t: \bar{y}_t \overline{\in} D\}, \quad \bar{\tau} = \min(\tau_x, \tau_y).$$

The quantity $\bar{\tau}$ does not depend on the future (see Note); therefore, the generalized solution $u(x)$ of problem (3) satisfies the relations

$$u(x) = Mu(\bar{x}_{\bar{\tau}}), \quad u(y) = Mu(\bar{y}_{\bar{\tau}})$$

(see $\lfloor 2\rfloor$). Let us estimate the difference $u(x) - u(y)$:

$$
\begin{aligned}
|u(x) - u(y)| &\leqslant M\,|u(\bar{x}_{\bar{\tau}}) - u(\bar{y}_{\bar{\tau}})| = \\
&= M\,|\psi(\bar{x}_{\tau_x}) - u(\bar{y}_{\tau_x})|\,\chi_x(\omega)) + M\,|u(\bar{x}_{\tau_y}) - \psi(\bar{y}_{\tau_y})|\,\chi_y(\omega) = \\
&= M\,|\psi(\bar{x}_{\tau_x}) - M_{\bar{y}_{\tau_x}}\psi(\bar{x}_\tau)|\,\chi_x(\omega) + M\,|M_{\bar{x}_{\tau_y}}^{\,\bar{z}}\psi(\tau_\tau) - \psi(\bar{y}_{\tau_y})|\,\chi_y(\omega),
\end{aligned}
\tag{9}
$$

where $\chi_x(\omega)$ is the characteristic function of the set $\{\omega: \bar{\tau} = \tau_x\}$, $\chi_y(\omega) = 1 - \chi_x(\omega)$. Let $a \in D$, $b \in \tilde{\Gamma}$. Then, if by d we denote the diameter of region D, we get

$$
\begin{aligned}
M_a|\psi(x_\tau) - \psi(b)| &\leqslant K_1 M_a|x_\tau - b|^\nu \leqslant K_1 d^\nu \cdot M_a \left|\frac{x_\tau - b}{d}\right|^\nu \leqslant \\
&\leqslant \begin{cases} K_1 d^\nu M_a \left|\dfrac{x_\tau - b}{d}\right|^{\lambda_1} = K_1 d^{\nu - \lambda_1} M_a|x_\tau - b|^{\lambda_1}, & \text{if } \lambda_1 < \gamma, \\[2mm] K_1 (M_a|x_\tau - b|^{\lambda_1})^{\nu/\lambda_1}, & \text{if } \lambda \geqslant \gamma. \end{cases}
\end{aligned}
\tag{10}
$$

Note. The non-negative random variable $\beta(\omega)$ is said to be independent of the future if the event $\{\omega: \beta(\omega) < t\}$ for any $t \geq 0$ belongs to a σ-algebra generated by the events $\{x_s(\omega) \in A\}$ for $s < t$, where A is a Borel set in R^n (see $\lfloor 2\rfloor$).

In the last inequality we made use of the fact that

$$(M|\xi|^{\nu})^{\lambda_i} \leqslant (M|\xi|^{\lambda_i})^{\nu}, \quad \text{if} \quad \lambda_i \geqslant \gamma.$$

By α we shall denote the characteristic operator of process X. It is verified that the function

$$v_1(x) = M_x|x_\tau - b|^{\lambda_i}$$

satisfies the equation $\alpha\, v_1 = 0$ in D and the boundary condition

$$v_1(x)|_{x \in \Gamma} = |x - b|^{\lambda_i}.$$

Since all points of the set $\widetilde{\Gamma}$ are $(\lambda_1, \lambda_2, h)$-regular, a neighborhood $U_h(b)$ will be found in which the barrier $v(x)$ is defined and $\alpha v(x) = Lv < 0$ in this neighborhood. We shall choose c_1 so great that $c_1 v(x) > v_1(x)$ on the boundary of the set $D \cap U_h(b)$. This can be done since the above-mentioned inequality is fulfilled on $\widetilde{\Gamma} \cap U_h(b)$. On the set $D \cap U_h(b)$ $\alpha\, (c_1 v - v_1) < 0$; therefore, by virtue of the maximum principle, everywhere in $D \cap U_h(b)$

$$v_1(x) = M_x|x_\tau - b|^{\lambda_i} < c_1 v(x) < \tilde{c}_2|x - b|^{\lambda_i}. \tag{11}$$

Obviously, if $\tilde{c}$ is chosen sufficiently great, we can consider that (11) is fulfilled everywhere in D.

We set

$$\lambda = \max\left(\lambda_2, \frac{\lambda_3 \gamma}{\lambda_4}\right).$$

From (10) and (11) it follows that

$$M_a|\psi(x_\tau) - \psi(b)| \leqslant c|a - b|^{\lambda}.$$

From the latter inequality and (9) we conclude that

$$|u(x) - u(y)| < c_3 M |x_\tau - \tilde{y}_\tau|^{\lambda}. \tag{12}$$

Let us now estimate the right-hand side of inequality (12). Since according to the condition of the theorem the trajectories leave the region at a uniform speed, then, according to lemma 4, it will be possible to find an $\alpha = \alpha_{L,D} > 0$ such that

$$P\{\bar{\tau} > t\} \leqslant \max_{x \in D} P_x\{\tau > t\} \leqslant c_3 e^{-\alpha t}.$$

Let $0 < \varkappa \leqslant 1$ and let $\chi_n(\omega)$ be the characteristic function of the set $\{\omega : n \leqslant \bar{\tau}(\omega) < n+1\}$. Since $\left|\dfrac{x_{\bar{\tau}} - y_{\bar{\tau}}}{d}\right| < 1$, we have

$$M\left|\frac{x_{\bar{\tau}} - \bar{y}_{\bar{\tau}}}{d}\right|^{\lambda} \leqslant d^{\lambda} M\left|\frac{x_{\bar{\tau}} - \bar{y}_{\bar{\tau}}}{d}\right|^{\varkappa\lambda} = d^{\lambda(1-\varkappa)} M\,|x_{\bar{\tau}} - \bar{y}_{\bar{\tau}}|^{\varkappa\lambda} \leqslant$$

$$\leqslant c_4 \sum_{n=0}^{\infty} M\,|x_{\bar{\tau}} - \bar{y}_{\bar{\tau}}|^{\varkappa\lambda} \chi_n(\omega) \leqslant c_4 \sum_n M\{\sup_{n \leqslant s \leqslant n+1} |x_s - \bar{y}_s|^{\varkappa\lambda} \chi_n(\omega)\} \leqslant$$

$$\leqslant c_4 \sum_n^{\infty} (M(\sup_{n \leqslant s \leqslant n+1} |x_s - y_s|)^{2\varkappa\lambda} \cdot P\{\bar{\tau} > n+1\})^{\frac{1}{2}} \leqslant$$

$$\leqslant c_5 \sum_{n=0}^{\infty} e^{-\frac{\alpha n}{2}} \cdot (M \sup_{s < n+1} |x_s - y_s|^2)^{\frac{\varkappa\lambda}{2}} \leqslant$$

$$\leqslant c_6 |x - y|^{\varkappa\lambda} \sum_{n=0}^{\infty} \beta_1(n) e^{-\frac{\alpha n}{2}} e^{\frac{\alpha_1 n \cdot \varkappa\lambda}{2}}. \tag{13}$$

In the latter inequality we used the corollary from lemma 5. Now, let $\varkappa$ be chosen in such a way that $\alpha_1 \lambda \varkappa < \alpha$. Then, the series in (13) converges and we obtain finally

$$|u(x) - u(y)| < c_6 |x - y|^{\varkappa\lambda}.$$

Lemma 5. **Let**

$$K = \max_{i,\,j,\,k;\,x \in R^n} \left\{\left|\frac{\partial \sigma_{ij}(x)}{\partial x_k}\right|,\quad \left|\frac{\partial b_i(x)}{\partial x_k}\right|\right\},$$

and let $\bar{x}_t$ **and** $\bar{y}_t$ **be the solutions of equation** (2) **with the initial conditions** $\bar{x}_0 = x$, $\bar{y}_0 = y$. **Then**

$$M\,|\bar{x}_t - \bar{y}_t|^{2m} \leqslant |x - y|^{2m} e^{\alpha_m t},$$

$$\alpha_m = 2m(m-1)K^2 n^2 + mn^3 K^2 + 2mnK.$$

- 19 -

<u>Proof</u>. Let us apply to the function $\rho^{2m}(\bar{x}_t, \bar{y}_t) = \left(\sum\limits_{i=1}^{n}(\bar{x}_t - \bar{y}_t^i)^2\right)^m$ the formula of K. Ito [7]:

$$\rho^{2m}(\bar{x}_t, \bar{y}_t) =$$

$$= \rho^{2m}(x, y) + \int\limits_0^t 2m\rho^{2m-2}(\bar{x}_s, \bar{y}_s)\left(\sum\limits_{i=1}^{n}(\bar{x}_s^i - \bar{y}_s^i)(b_i(\bar{x}_s) - b_i(\bar{y}_s))\right)ds +$$

$$+ \int\limits_0^t 2m\rho^{2m-2}(\bar{x}_s, \bar{y}_s)\left(\sum\limits_{i=1}^{n}(\bar{x}_s^i - \bar{y}_s^i)\sum\limits_{j=1}^{n}[\sigma_{ij}(\bar{x}_s) - \sigma_{ij}(\bar{y}_s)]d\xi_s^j\right) +$$

$$+ \frac{1}{2}\int\limits_0^t 4m(m-1)\rho^{2m-4}(\bar{x}_s, \bar{y}_s)((\sigma(\bar{x}_s) - \sigma(\bar{y}_s))(\bar{x}_s - \bar{y}_s),$$

$$(\sigma(\bar{x}_s) - \sigma(\bar{y}_s))(\bar{x}_s - \bar{y}_s))ds +$$

$$+ \frac{1}{2}\int\limits_0^t 2m\rho^{2m-2}(\bar{x}_s, \bar{y}_s)\sum\limits_{i, k=1}^{n}(\sigma_{ik}(\bar{x}_s) - \sigma_{ik}(\bar{y}_s))^2 ds. \tag{14}$$

From (14), using the properties of stochastic integrals and the boundedness of the derivatives of the coefficients, we obtain the inequality

$$M\rho^{2m}(\bar{x}_t, \bar{y}_t) \leqslant \rho^{2m}(x, y) + \alpha_m\int\limits_0^t M\rho^{2m}(\bar{x}_s, \bar{y}_s)ds.$$

The statement of the lemma follows from the latter inequality (see, for example, [18]).

<u>Corollary</u>. For any $T > 0$ the following inequality is valid:

$$M\sup_{t<T}|\bar{x}_t - \bar{y}_t|^2 \leqslant c\left[|x + y|^2 + M\sup_{t<T}\left|\int\limits_0^t[\sigma(\bar{x}_s) - \sigma(\bar{y}_s)]d\xi_s\right|^2 +\right.$$

$$+ M\left(\int\limits_0^T|b(\bar{x}_s) - b(\bar{y}_s)|ds\right)^2 \leqslant c\left[|x - y|^2 + \int\limits_0^T M|\sigma(\bar{x}_s) - \sigma(\bar{y}_s)|^2 ds +\right.$$

$$+ T\int\limits_0^T M|x_s - y_s|^2 ds \leqslant \beta(t)e^{\alpha t}.$$

In estimating the stochastic integral we made use of the fact that

$$\left|\int\limits_0^t[\sigma(x_s) - \sigma(y_s)]d\xi_s\right|^2$$

is a semi-martingale and, consequently, according to theorem 3.4 of Chapt. 7 [3]

$$M \sup_{t < \tau} \left| \int_0^t [\sigma(x_s) - \sigma(y_s)] d\xi_s \right|^2 \leqslant \int_0^T M |\sigma(\bar{x}_s) - \sigma(\bar{y}_s)|^2 ds.$$

Note that $\beta(t)$ increases with respect to t not faster than linearly.

4. A Priori Estimates of Derivatives of the Generalized Solution

Theorem 2. Let us assume that $\sigma_{ij}(x)$, $b_i(x) \in C^1(R^n)$, that all points of the set $\tilde{\Gamma}$ are uniformly normally regular and that $\psi(x) = v(x)|_{\tilde{\Gamma}}$, where $v(x) \in C^2(R^n)$. Then, the generalized solution satisfies the Lipschitz condition if only $\alpha_{L,D} > \alpha_1$.

Proof. We shall use the same notation as that used in the proof of theorem 1. From (9) it follows that, in order to prove theorem 2, it is sufficient to estimate the terms occurring on the right-hand side of inequality (9). In order to do this, we shall apply to the function v(x) the formula of K. Ito

$$v(x_t) = v(x) + \int_0^t (\text{grad } v(x_s), \sigma(x_s) d\xi_s) + \int_0^t Lv(x_s) ds.$$

From the latter equality it follows (see Note) that

$$|M_z v(x_\tau) - v(z)| = \left| M \int_0^\tau Lv(x_s) ds \right| < \|Lv(x)\| M_z \tau. \tag{15}$$

For the right-hand side of inequality (9) we obtain the estimate

$$M |\psi(x_{\tau_x}) - M_{\bar{y}_{\tau_x}} \psi(x_\tau)| \chi_x(\omega) \leqslant M \chi_x(\omega) |v(x_{\tau_x}) - M_{\bar{y}_{\tau_x}} v(x_\tau)| \leqslant$$
$$\leqslant M \chi_x(\omega) |v(x_{\tau_x}) - v(\bar{y}_{\tau_x})| + M \chi_x(\omega) |v(\bar{y}_{\tau_x}) - M_{\bar{y}_{\tau_x}} v(x_\tau)| \leqslant$$
$$\leqslant \|\text{grad } v(x)\| M |x_{\tau_x} - \bar{y}_{\tau_x}| \chi_x(\omega) + \|Lv\| M (M_{\bar{y}_{\tau_x}} \tau) \chi_x(\omega) \leqslant c (\|\text{grad } v(x)\| +$$
$$+ \|Lv\|) M |x_{\tau_x} - \bar{y}_{\tau_x}| \chi_x(\omega).$$

$$\tag{16}$$

Note: By $\|f(x)\|$ we denote $\max_{x \in R^n} |f(x)|$.

In the derivation of this inequality we used the strict Markov property of process X and the uniform normal regularity of the points of set $\tilde{\Gamma}$. A similar inequality also holds for the second term on the right-hand side of (9).

From (9) and (16) follows the inequality

$$|u(x) - u(y)| < c_2 M |\bar{x}_{\bar{\tau}} - \bar{y}_{\bar{\tau}}|. \qquad (17)$$

Further, after using the corollary of lemma 5 we get

$$M|\bar{x}_{\tau} - \bar{y}_{\tau}| \leqslant \sum_{n=0}^{\infty} M \sup_{n \leqslant s < n+1} |\bar{x}_s - \bar{y}_s| \chi_n(\omega) \leqslant$$

$$\leqslant \sum_{n=0}^{\infty} (M \sup_{s < n+1} |\bar{x}_s - \bar{y}_s|^2 \cdot M\chi_n(\omega))^{1/2} \leqslant$$

$$< c_3 \left(\sum_{n}^{\infty} \beta_n e^{\frac{\alpha}{2}n} e^{-\frac{n}{2}\alpha_{L,D}} \right) |x - y| < c|x - y|. \qquad (18)$$

In the latter inequality

$$\sum \beta_n e^{-\frac{n}{2}(\alpha_{L,D} - \alpha_1)} < \infty,$$

since $\alpha_{L,D} > \alpha_1$ according to the condition of the theorem. From (17) and (18) follows the statement of theorem 2.

Let us now consider the estimates of the higher derivatives of the generalized solution. We shall define the m-th difference for the function f(x) by means of the equations

$$\delta_1^h f(x) = f(x + h) - f(x),$$

$$\delta_m^{h_1, \ldots, h_m} f(x) = \delta_{m-1}^{h_1, \ldots, h_{m-1}} f(x + h_m) - \delta_{m-1}^{h_1, \ldots, h_{m-1}} f(x).$$

If the function f(x) is sufficiently smooth, by selecting the increments $h_1, \ldots, h_m \in R^n$, $|h_i| = h$ it is possible to obtain that

$$\lim \frac{1}{h^m} \delta_m^{h_1, \ldots, h_m} f(x) = \frac{\partial^m f(x)}{\partial x_1^{i_1} \ldots \partial x_n^{i_n}}.$$

- 22 -

for any $l_1, \ldots, l_n$ summing to m. By $H = H(x, h_1, \ldots, h_m)$ we shall denote the set of points $x + h_{i_1} + \ldots + h_{i_r}$ for $i_1, \ldots, i_r \leq m$, $0 \leq r \leq m$; by H_f - the convex hull of the values of function $f(y)$ for $y \in H$.

We shall use the following simple inequality:

$$|\delta_m u(f(x))| \leq |(\operatorname{grad} u(x), \delta_m f(x))| + \|D_m u\|_{H_f} \cdot$$

$$\sum_{\substack{\Sigma m_i l_i = m, \\ l_i < n}} |\delta_{l_i} f(x)|^{m_1} \ldots |\delta_{l_i} f(x)|^{m_r}, \tag{19}$$

where

$$\|D_m u\|_A = \max_{\substack{\Sigma m_i = k \leq m, \\ x \in A}} \left| \frac{\partial^k u(x)}{\partial x_1^{m_1} \ldots \partial x_n^{m_n}} \right|$$

Let us define the sequence of the numbers $\alpha_{k,m}$ by means of the following relations:

$$a_{1,m} = a_m = (2m(m-1)n^3 + mn^3)K^2 + 2mnK$$
$$a_{k,1} = 2Kn + 2K^2 n^3 + 1 + a_{k-1,k} \quad \text{for} \quad k > 1,$$
$$a_{k,m} = 2m(nK + 1) + m(2n^3 K^2 + 1) + 2m(m-1)n^3 K^2 + 1 + a_{k-1,km}$$
$$\text{for} \quad k > 1.$$

It is easy to check that $\alpha_{k,m}$ grows monotonically with respect to each index.

Lemma 6. For all integral k and m the following inequality* is valid:

$$M(\delta_h^{h_1 \ldots h_k} x_t^x)^{2m} \leq C_{k,m} h^{2km} e^{a_{k,m} t}, \quad h = |h_i|.$$

The proof of this lemma is carried out by induction. For k = 1, lemma 6 reduces to lemma 5. We will show that the statement of the lemma is preserved when k is increased by unity. Since δ_k is a linear operator, we have

$$\delta_h x_t^x = \int_0^t \delta_h \sigma(x_s^x) d\xi_s + \int_0^t \delta_h b(x_s^x) ds.$$

---

* The operator δ_k is applied to $x_t^x(\omega)$ as in the function of the initial point $x = x_0^x(\omega)$.

Let us apply the formula of K. Ito to the function $\mu_k^2(t) = \sum_1^n (\delta_k x_t^x)^2$:

$$\mu_k^2(t) = \mu_k^2(0) + 2 \int_0^t \sum_{i=1}^n \left(\delta_k x_s^i \sum_{j=1}^n \delta_k \sigma_{ij}(x_s^x) d\xi_{bs}^j \right) +$$
$$+ 2 \int_0^t \sum_{i=1}^n \delta_k x_s^i \delta_k b_i(x_s^x) ds + \int_0^t \sum_{i,j=1}^n (\delta_k \sigma_{ij}(x_s^x))^2 ds.$$

$$(20)$$

From (19) and (20) we get

$$m_k(t) = M\mu_k^2(t) \leqslant m_k(0) + 2 \int_0^t \sum_{i=1}^n |\delta_k x_s^i| \left[|(\operatorname{grad} b_i(x_s), \delta_k x_s^x)| + \right.$$

$$+ \|D_k u\| \sum_{\substack{\Sigma m_i l_i = k \\ l_i < k}} |\delta_{l_1} x_s^x|^{m_1} \ldots |\delta_{l_r} x_s^x|^{m_r} \right] ds + 2 \int_0^t \left[\sum_{i,j=1}^n |(\operatorname{grad} \sigma_{ij}(x_s), \delta_k x_s^x)| + \right.$$

$$+ \|D_k u\| \sum_{\substack{\Sigma l_i m_i = k, \\ l_i < k}} |\delta_{l_1} x_s^x|^{m_1} \ldots |\delta_{l_r} x_s^x|^{m_r} \right] ds \leqslant$$

$$\leqslant m_k(0) + (2Kn + 1) \int_0^t m_k(s) ds + 2n^3 K^2 \int_0^t m_k(s) ds +$$

$$+ c \int_0^t \sum_{\substack{\Sigma l_i m_i = k, \\ l_i < k}} M |\delta_{l_1} x_s^x|^{2m_1} \ldots |\delta_{l_r} x_s^x|^{2m_r} ds.$$

$$(21)$$

We shall now make use of the fact that

$$M |\delta_{l_1} x_s|^{2m_1} \ldots |\delta_{l_r} x_s^x|^{2m_r} \leqslant M \sum_{l < k} \delta_l^{2k/l} x_s \leqslant \sum_{l < k} (M \delta_l^{2k} x_s^x)^{1/l}.$$

From (21) we get

$$m_k(t) \leqslant m_k(0) + (2Kn + 2K^2 n^3 + 1) \int_0^t m_k(s) ds + c \int_0^t \sum_{l < k} (M|\delta_l x_s^x|^{2k})^{1/l} ds.$$

According to the assumption of induction for $\ell < k$

$$M|\delta_l x_t^x|^{2k} \leqslant C_{l,k} e^{\alpha_{l,k} t} \cdot h^{2lk} \leqslant C_{k-1,k} e^{\alpha_{k-1,k} t} h^{2lk},$$

since $\alpha_{\ell,k}$ increases with increasing ℓ. From (22) follows the inequality

$$m_k(t) \leqslant m_k(0) + (2Kn + 2K^2n^3 + 1) \int\limits_0^t m_k(s)\,ds + C_{k-1,k} e^{\alpha_{k-1,k}t} h^{2k}.$$

From the last inequality we conclude that

$$m_k(t) \leqslant C_{k,1} e^{\alpha_{k,1}t} \cdot h^{2k}.$$

In order to check the validity of the lemma when the second index increases, it is necessary to apply Ito's formula to the function $\left(\sum\limits_i (\delta_k x_t^i)^2 \right)^m$. This calculation is completely analogous to lemma 5 and we omit it.

Lemma 7. **Let** $f(s,\omega)$ **be a bounded measurable function. Then,**

$$M\left(\int\limits_0^T f(s,\omega)\,d\xi_s \right)^{2m} \leqslant c_m T^{m-1} \int\limits_0^T M|f(s,\omega)|^{2m}\,ds.$$

Proof. Applying Ito's formula to the function $\left(\int\limits_0^T f(s,\omega)\,d\xi_s \right)^{2m}$ we get

$$g_m(T) = M\left(\int\limits_0^T f(s,\omega)\,d\xi_s \right)^{2m} = m(2m-1) \int\limits_0^T M\left(\int\limits_0^t f(s,\omega)\,d\xi_s \right)^{2(m-1)} f^2(t)\,dt.$$

From this formula it follows that the function $g_m(T)$ increases monotonically with respect to T. From Hölder's inequality, taking monotonicity into account, it follows that

$$M\left(\int\limits_0^T f(s,\omega)\,d\xi_s \right)^{2m} \leqslant$$

$$\leqslant m(2m-1) \left[\int\limits_0^T M\left(\int\limits_0^t f(s,\omega)\,d\xi_s \right)^{2m}\,dt \right]^{\frac{m-1}{m}} \left[\int\limits_0^T M f^{2m}(t,\omega)\,dt \right]^{1/m} \leqslant$$

$$\leqslant m(2m-1) T^{\frac{m-1}{m}} \left[M\left(\int\limits_0^T f(s,\omega)\,d\xi_s \right)^{2m} \right]^{\frac{m-1}{m}} \left[\int\limits_0^T M f^{2m}(t,\omega)\,dt \right]^{1/m}.$$

From the last inequality we conclude that

$$M\left[\int_0^T f(s,\omega)\,d\xi_s\right]^{2m} \leqslant [m(2m-1)]^m T^{m-1}\int_0^T Mf^{2m}(s,\omega)\,ds.$$

Lemma 8. For any positive integral k and m the following inequality holds:

$$M\sup_{s<t}|\delta_h^{h_1,\ldots,h_k}x_s^x|^{2m} < \tilde{C}_{h,m}(t)h^{2km}\cdot e^{\alpha_{h,m}t},$$

where $\tilde{C}_{k,m}(t)$ increases with increasing t not faster than t^{2m-1}, $|h_i| = h$.

This lemma is proven in the same way as the corollary of lemma 5. It is only necessary to make use of the fact that $\left(\int_0^t f(s,\omega)\,d\xi_s\right)^{2m}$ is a semi-martingale and to use lemma 7 to estimate the 2m-th power of the stochastic integral. Let

$$\tau^\nu(\omega) = \inf\{t : x_t^\nu \bar{\in} D\}, \quad \bar{\tau} = \min_{\nu\in H(x,h_1,\ldots,h_k)}\tau^\nu.$$

It is obvious that

$$P\{\bar{\tau} > t\} \leqslant P_x\{\tau > t\} \leqslant ce^{-\alpha_{L,D}t}.$$

Lemma 9. If $\alpha_{k,m} < \alpha_{L,D}$ и $|h_i| = h$, then

$$M|\delta_h^{h_1,\ldots,h_k}x_\tau^x|^m \leqslant h^{km}\tilde{C}.$$

Proof. Let $\chi_n(\omega)$ be the characteristic function of the set $\{\omega : n \leqslant \bar{\tau} \, n + 1\}$. Then, using the preceding lemma we obtain

$$M|\delta_h x_\tau^x|^m \leqslant \sum_{n=0}^\infty M\sup_{n<s<n+1}|\delta_h x_s^x|^m\chi_n(\omega) \leqslant \sum_0^\infty (M\sup_{s<n+1}|\delta_h x_s^x|^{2m}\cdot M\chi_n(\omega))^{1/2} \leqslant$$

$$\leqslant \sum_{n=0}^\infty C_{k,m}^{(n+1)}\cdot e^{\frac{n}{2}\alpha_{k,m}}\cdot e^{-\frac{n}{2}\alpha_{L,D}}h^{km} \leqslant \tilde{C}h^{km}.$$

The series in the last inequality converges because $C_{k,m}^{(n)}$ increases exponentially with respect to n and $\alpha_{L,D} > \alpha_{k,m}$ according to the condition of the lemma.

Lemma 10. _Assume that_ $\alpha_{2k,2k} < \alpha_{L,D}$. _Then,_

$$\|D_\lambda u\|_D < C\|D_\lambda u\|_{\bar\Gamma}.$$

Proof. Let us choose $h_1,\ldots,h_m$, $|h_i| = h$, $m \geq k$, in such a way that

$$\lim \frac{1}{h^m} \delta_m^{h_1,\ldots,h_m} u(x) = \frac{\partial^m u}{\partial x_1^{l_1}\ldots\partial x_n^{l_n}}.$$

Let

$$\bar\tau = \min_{v\in H(x,h_1,\ldots,h_m)} \tau^v(\omega).$$

The quantity $\bar\tau$ does not depend on the future, and therefore $u(x) = Mu(x_{\bar\tau}^x)$. Consequently, for $\delta_h u(x)$ we get the inequality

$$|\delta_m u(x)|^2 \leqslant (M|\delta_m u(x_{\bar\tau}^x)|)^2 \leqslant \Big[\|D_m u\|_{U_{\epsilon_1}(\bar\Gamma)}^2 M\Big(|\delta_m x_{\bar\tau}^x|^2 +$$

$$+ \sum_{\substack{\Sigma l_i m_i=m \\ l_i<m}} |\delta_{l_1} x_{\bar\tau}^x|^{2m_1}\ldots|\delta_{l_r} x_{\bar\tau}^x|^{2m_r}\Big)(1-\epsilon_2) + \epsilon_2 \|D_m u\|_D^2 M\left(|\delta_m x_{\bar\tau}^x|^2 +\right.$$

$$\left. + \sum_{\substack{\Sigma l_i m_i=m, \\ l_i<m}} |\delta_{l_1} x_{\bar\tau}^x|^{2m_1}\ldots|\delta_{l_r} x_{\bar\tau}^x|^{2m_r}\right)\Big] C_1. \tag{22}$$

Here we have used the fact that, for sufficiently small h, the points $x_{\bar\tau}^y$ ($y \in H(x, h_1,\ldots,h_m)$) lie in the ϵ_1-neighborhood of $\bar\Gamma$ with a probability greater than $1 - \epsilon_2$. With the aid of lemma 9 we can derive the inequalities

$$M|\delta_m x_{\bar\tau}^x|)^2 < \bar{C}h^{2m}, \quad M|\delta_{l_1} x_{\bar\tau}^x|^{2m_1}\ldots|\delta_{l_r} x_{\bar\tau}^x|^{2m_r} \leqslant \sum_{l<m} (M|\delta_l x_{\bar\tau}^x|^{2lm})^{1/l} \leqslant h^{2m}\widetilde{C}. \tag{23}$$

Here we make use of the fact that the sequence $\alpha_{\ell,r}$ is monotonic with respect to each index and also of the condition $\alpha_{L,D} > \alpha_{2k,2k}$.

From (22) and (23) it follows that

$$\left|\frac{1}{h^m}\delta_m u(x)\right|^2 \leqslant C\left(\|D_m u\|_{U_{\epsilon_1}(\bar\Gamma)}^2 \cdot (1-\epsilon_2) + \epsilon_2\|D_m u\|_D\right).$$

Now let h tend to zero. Together with h, ϵ_1 and ϵ_2 also tend to zero. From the last inequality we obtain the statement of the lemma.

Theorem 3. *Let* $\sigma_{ij}(x)$, $b_i(x) \in C^{(k)}(R^n)$, $\psi(x) \in C(\widetilde{\Gamma})$, *let the operator* L *not degenerate in the neighborhood of the set* $\widetilde{\Gamma}$, *and let* $\alpha_{L,D} > \alpha_{2k,2k}$. *Then, the generalized solution of problem* (3) *has* k - 1 *continuous derivatives and the* (k - 1)-*th derivatives satisfy the Lipschitz condition.*

Proof. First, let us assume that $\widetilde{\Gamma}$ is a smooth manifold of class C^k and $\psi(x) \in C^k(\widetilde{\Gamma})$. We shall examine the operator L^ϵ, whose coefficients are determined by the equalities

$$\{ \overset{\bullet}{a}_{ij}(x) \} = (\sigma(x) + \epsilon E \cdot \varphi(x)) \cdot (\sigma^*(x) + \epsilon\varphi(x)E), \quad b_i^\epsilon(x) = b_i(x).$$

We shall choose the infinitely differentiable function $\varphi(x)$ in such a way that it vanishes in the neighborhood of the boundary and is positive outside the δ-neighborhood of the boundary. As follows from [18], there exists a unique generalized solution of the problem

$$L^\epsilon u^\epsilon(x) = 0 \quad \text{for} \quad x \in D; \quad u^\epsilon(x)|_{\widetilde{\Gamma}} = \psi(x).$$

This generalized solution is continuous. The operator L^ϵ does not degenerate outside the δ-neighborhood of the boundary, and therefore $u^\epsilon(x) \in C^{(k)}(D \setminus U_\delta(\Gamma))$. Let

$$\alpha_{L,D} - \alpha_{2k,2k} = \lambda.$$

Then, for a sufficiently small ϵ

$$|\alpha_{L^\epsilon,D} - \alpha_{L,D}| < \frac{\lambda}{2}$$

and the coefficient $\alpha_{2k,2k}^\epsilon$ for operator L^ϵ differs from $\alpha_{2k,2k}$ by not more than $\lambda/2$. Therefore, $\alpha_{L,D}^\epsilon - \alpha_{2k,2k}^\epsilon > 0$ and, consequently, the first k-derivatives of function $u^\epsilon(x)$ are estimated according to lemma 10 in terms of derivatives on the boundary $\widetilde{\Gamma}$.

In the neighborhood of $\widetilde{\Gamma}$, the operator L^ϵ does not depend on ϵ, $|u^\epsilon(x)| < \max\limits_{x \in \widetilde{\Gamma}} |\psi(x)|$. so that $\|D_m u^\epsilon\|_{\widetilde{\Gamma}}$ is estimated uniformly with respect to ϵ, and, therefore, according to lemma 10, the derivatives are estimated uniformly with respect to ϵ also inside region $D \backslash U_\delta(\Gamma)$. It is proven (see [21]) that

$$\lim_{\epsilon \to 0} u^\epsilon(x) = u(x)$$

uniformly with respect to $x \in D$.

From the last statement and the _a priori_ estimates uniform with respect to ϵ the statement of the theorem can be derived in a standard manner. If $\widetilde{\Gamma} \in C^k$, then it is necessary to replace $\widetilde{\Gamma}$ by the surface $\widetilde{\Gamma} \in C^k$ lying in the region where operator L is non-degenerate. The function $u(x)$ on $\widetilde{\widetilde{\Gamma}}$ will be sufficiently smooth. This follows from the usual intrinisic estimates for uniformly elliptic equations.

Theorem 4. Assume that $\sigma_{ij}(x)$, $b_i(x) \in C^k(R^n)$, that the boundary Γ of region D has the direction cosines $\{n_i(x)\}$ of class C^k, that $\psi(x) \in C^k(\Gamma)$ and that $\alpha_{L,D} > \alpha_{2k,2k}$. Moreover, let the diffusion along the normal to the boundary $(\Sigma a_{ij}(x)n_i(x)n_j(x))$ be different from zero on the closure of $\widetilde{\Gamma}$, let it be equal to zero on $\Gamma \backslash \widetilde{\Gamma}$, and let the vector $b(x)$ on $\widetilde{\Gamma}$ be directed towards the exterior of region D. Then, $u(x) \in C^{k-1}(D)$ and the $(k - 1)$-th derivatives satisfy the Lipschitz condition.

Proof. If $\Gamma = \widetilde{\Gamma}$, then the statement of the theorem follows from lemma 10 and lemma 1.4 of reference [12]. If $\Gamma \backslash \widetilde{\Gamma}$ is not empty, we extend the operator L across $\widetilde{\Gamma}$ into region $D \cup \widetilde{D}$ in such a way that everywhere on the boundary $\widetilde{\Gamma} \cup \widetilde{\widetilde{\Gamma}}$ of the region $D \cup \widetilde{D}$ the extended operator $\widetilde{L}$ has a non-zero diffusion along the normal and the maximum of the modulus of the derivatives of the coefficients does not increase.

We extend the function $\psi(x)$ onto $\widetilde{\widetilde{\Gamma}}$ in an arbitrary smooth manner. This is possible since $\rho(\widetilde{\Gamma},\widetilde{\widetilde{\Gamma}}) > 0$. For the operator $\widetilde{L}$ diffusion along the normal does not degenerate,

$$|\alpha_{\widetilde{L},D\cup\widetilde{D}} - \alpha_{L,D}| < \frac{\alpha_{L,D} - \alpha_{2k,2k}}{2}$$

for a sufficiently small extension, the constants $\alpha_{2k,2k}$ for operator $\widetilde{L}$ will remain the same and, therefore, as was noted at the beginning of the proof, the assertion of the theorem is fulfilled for the solution $\widetilde{u}(x)$ of the problem in $D \cup \widetilde{D}$. Therefore, the theorem is also fulfilled for the solution of problem (3) since the solutions of the extended problem and of problem (3) coincide in the region D. The latter assertion follows from the fact that the trajectories of the process governed by the extended operator $\widetilde{L}$ with the probability 1, originating from points $x \in D$, leave D across $\widetilde{\Gamma}$ since $\Gamma \backslash \widetilde{\Gamma}$ is a repelling boundary.

5. Some Remarks

1. The parabolic equation

$$\frac{\partial u}{\partial t} = \frac{1}{2} \sum a_{ij}(t,x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum b_i(t,x) \frac{\partial u}{\partial x_i} + c(t,x)u(t,x) =$$
$$= Lu + c(t,x)u$$

can be considered as a degenerating elliptic equation. The processes governed by the operator $\mathcal{L} = -\frac{\partial}{\partial t} + L$ are remarkable in that along one of the coordinates (along t) a determinate motion with a unit velocity takes place in the direction of the plane t = 0 (if the original point is in the region t > 0). Hence, it follows that the generalized solution of the problem $\mathcal{L}u = 0$ in the region lying above the plane $t = t_0$ is unique for some t_0. In particular, the generalized solution of the Cauchy problem and of the first boundary value problem in a cylinder is always unique. If the region in which the equation $\mathcal{L}u = 0$ is considered lies above the plane $t = t_0$, then, starting from any point lying below the plane t = T, the

- 30 -

trajectories will reach the boundary not later than in a time $T - t_o$. Since the trajectories reach the boundary in finite time, the parameter $\alpha_{\mathcal{L},D}$ can be considered equal to infinity for the region lying above the plane $t = t_o$. Thus, in the case of a boundary value problem for operator $\mathcal{L}$ in the region lying above the plane $t = t_o$, the smoothness of the generalized solution depends only on boundary estimates if only the coefficients of the equation are sufficiently smooth. In particular, in the case of the Cauchy problem for an equation with smooth coefficients, the same smoothness as in the initial function is transferred inside the region.

2. In [19] the problem with a directional derivative was studied for the equation Lu = f(x). The following theorem is derived from the results obtained in this article and reference [19].

Theorem 5. In the region D with a smooth boundary Γ let us consider the problem

$$Lu = f(x), \quad \frac{\partial u}{\partial l(x)}\bigg|_{\Gamma} = 0.$$

(24)

It is assumed that the coefficients of the operator and the field $l(x)$ belong to class C^3, the field $l(x)$ does not touch the boundary, the operator L does not degenerate in the neighborhood of Γ, and the trajectories of process X reach the boundary at a uniformly rapid rate. Then, there exists in D the unique probability measure $\mu(\cdot)$, namely a solution of the conjugate problem $\overrightarrow{L}^*\mu = 0$, such that problem (24) has a continuous generalized solution for any continuous f(x) for which $\int_D f(x)\mu(dx) = 0$. This generalized solution is unique with an accuracy up to the constant term. If $\sigma_{ij}(x)$, $b_i(x)$, $f(x) \in C^m(R^n)$ and $\alpha_{L,D} > \alpha_{m,m}$, the solution has

- 31 -

partial derivatives up to order m - 1 _inclusively and the_ m-_th derivatives satisfy_
the Lipschitz condition.

The definition of the generalized solution of problem (24) can be found in
[19], where an explanation of what L^* is, is also given. If we discard the assump-
tion of non-degeneracy near the boundary and we assume only that diffusion along
the normal is different from zero, then the solution of problem (24) will, generally
speaking, not be unique. If we consider the problem for the operator Lu - c(x)u = 0,
c(x) > c > 0, we can limit ourselves to the assumption of non-degeneracy along the
normal. In this case, a generalized solution exists for any right-hand side ,f C.
This solution is unique without any assumptions on a uniformly rapid exit from the
region. By making use of the results given in [19] it is also possible to study
the smoothness of the generalized solution of the second mixed problem for a para-
bolic equation with degeneration.

3. Let us consider the equation Lu(x) - c(x)u(x) = 0 in the region D with
the boundary conditions $u|_{\widetilde{\Gamma}} = \psi(x)$, where $\Gamma \backslash \widetilde{\Gamma}$ is a repelling set for operator L and
the points of $\widetilde{\Gamma}$ are assumed to be uniformly and normally regular. If, as was done
before, we denote by $\lambda = \{x_t, P_x\}$ the process which is governed by operator L, then
the generalized solution of the stated problem is given by the formula

$$u(x) = M_x \psi(x_\tau) \exp\left\{ - \int_0^\tau c(x_s) ds \right\}.$$

If c(x) ÷ 0 the written mathematical expectation exists and, consequently, a gen-
eralized solution also exists. It is proven that the function u(x) takes on boundary
conditions at normally regular points of the boundary and that the generalized

- 32 -

solution is unique if $c(x) > c > 0$ in D without any assumptions whatsoever concerning the exit of trajectories from the region.

Let us consider the question of smoothness of the function $u(x)$. Let at first $c(x) = c > 0$. Then, $u(x) = M_x e^{-c\beta} u(x_\beta)$ for any $\beta(\omega)$ not dependent on the future.

In particular, if $\beta = \bar{\tau}$ (see lemma 9), then

$$|\delta_k u(x)| = |Me^{-c\bar{\tau}}\delta_k u(x_{\bar{\tau}}^x)| \leqslant c_1 Me^{c\bar{\tau}}|\delta_k x_{\bar{\tau}}^x|^k \leqslant$$
$$\leqslant c_1 \sum_{n=1}^{\infty} e^{-c(n-1)} M \max_{s \leqslant n}|\delta_k x_s|^k \cdot \chi_n(\omega) \leqslant$$
$$\leqslant c_2 \sum \widetilde{C}_{k,k}(n) e^{n(a_{2k},\, 2k^{-a}L,\, D^{-c})} \cdot h^k.$$

Consequently, for the existence of an estimate of the k-th derivative it is sufficient that $a_{2k,2k} < a_{L,D} + c$. Thus, choosing c sufficiently large, it is possible to achieve an arbitrarily high smoothness if, of course, the coefficients are sufficiently smooth and there are boundary estimates.

The case when $c(x) > c > 0$ is reduced to the one examined above by dividing the equation by $c(x)/c$.

4. The conditions reported here for the existence of derivatives do not have a local character: smoothness at the point x depends, in general, on the geometry of the region and the behavior of the coefficients of the operator in the entire region. Simple examples show that such conditions reflect the actual situation in the class of all degenerating equations. However, special classes of degenerating equations do exist in which smoothness has a local character. For example, the

equation of Brownian motion with inertia is such an equation. Definite success in the study of such classes has been achieved in references [13] and [16].

5. In conclusion, we should note that the methods used here can be utilized for the study of degenerating quasilinear equations. These problems are considered in references [23,24].

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13. ABSTRACT

In this study it is proven that the generalized solution of boundary-value problems for degenerating second-order equations satisfies Holder's boundary condition under broad assumptions. An example is given, showing that a greater smoothness of the solutions cannot be achieved only by increasing the smoothness of the data given in the problem. Conditions for the existence of derivatives of the generalized solution are clarified. The investigation is carried out by means of probability methods.

1 4.

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